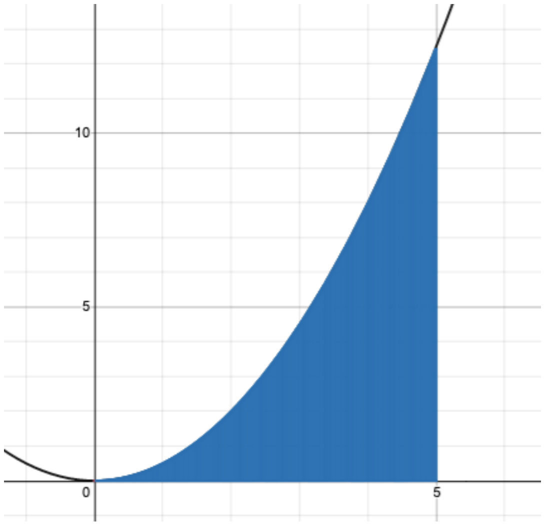
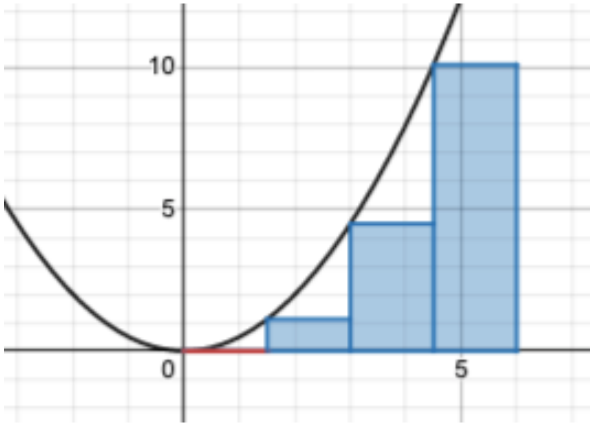
What is Calculus? Motivating the Need for a "Limit"

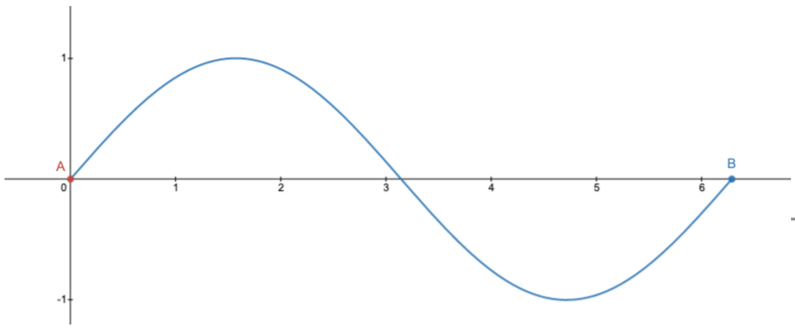
Consider the following problems.

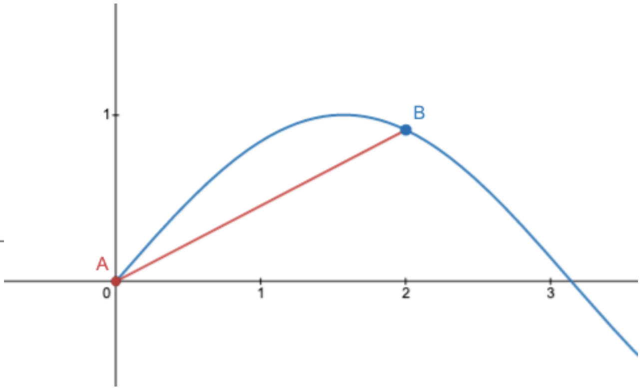
1) Area of Rectangle vs. Area Under a Curve
https://www.desmos.com/calculator/pesw8ofqvi (Animation Links in Canvas)



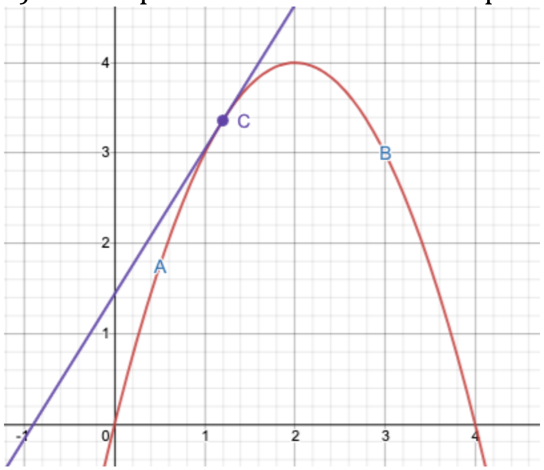


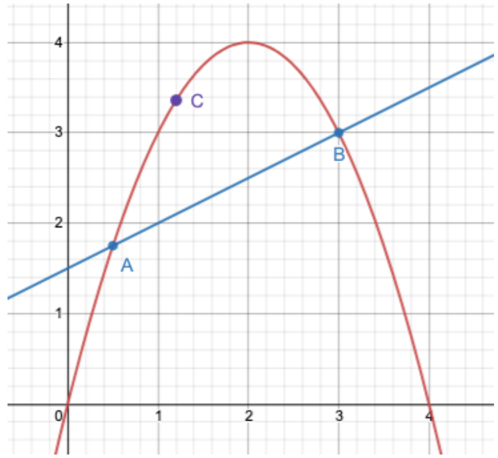
2) Length of Line Segment vs Length of Curve
https://www.desmos.com/calculator/6zdb5tjl6d





3) Slope of a Secant Line vs. Slope of a Tangent Line (Section 1.4 and 2.1).





4) Average Velocity vs. Instantaneous Velocity. (Section 1.4 and 2.1)

Each of these problems can be thought of as being connected to a simpler Precalculus problem. What we aim to do in Calculus is to use what we have learned in Precalculus and extend the simple problems into something more complex by breaking it into small pieces. In each of these cases, we need to use the concept of "getting closer and closer", that is, Limits.

1.5 and 1.7: The Limit

The idea of numbers "approaching a number" is fundamental to all of calculus. In this unit we will consider:

1) What do we formally mean by limit?
2) How do we compute the value of a limit?
3) How we can use the limit concept to compute instantaneous rate of change.

<u>The Limit Idea</u>

Example: Consider $f(x) = \dfrac{x-2}{x^2-4}$. What is $f(2)$? ______________________

What values does f take on *near* x=2?

1) <u>Numerical Approach</u>.

If x approaches 2 from the right (   )

x_1	$f(x_1)$
3	0.2
2.5	0.22222222
2.1	0.24390244
2.05	0.24691358
2.01	0.24937656
2.001	0.24993752

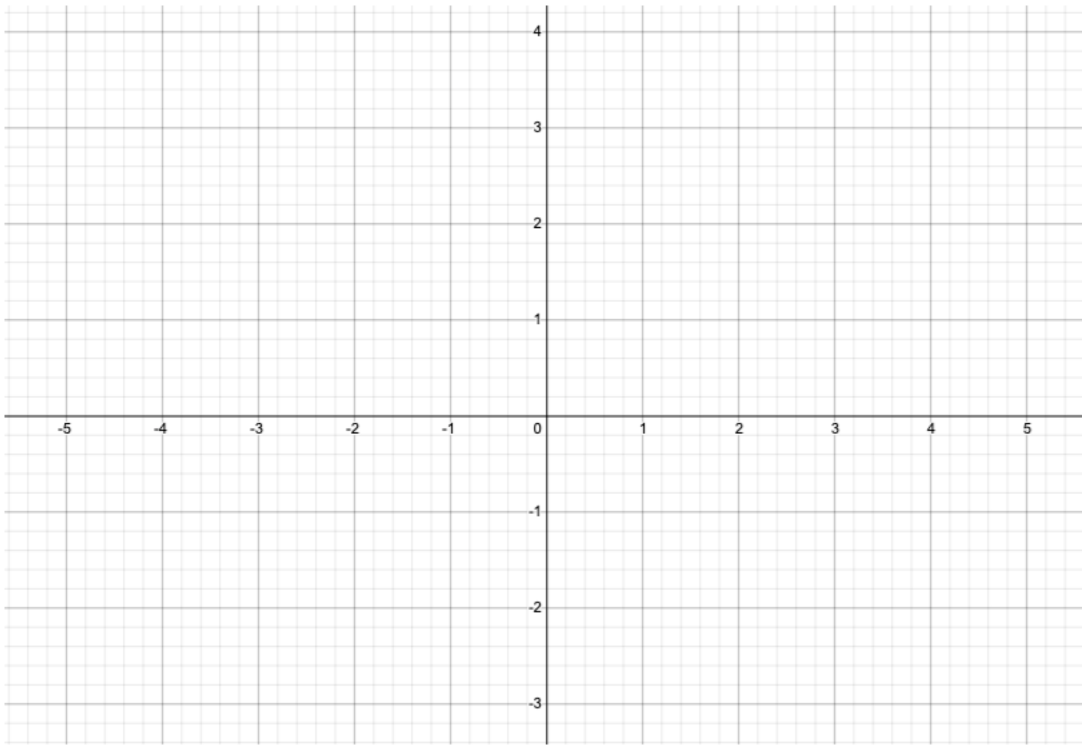
If x approaches 2 from the left (   )

x_1	$f(x_1)$

So we might say that as x gets close to 2, f(x) get close to ________

NOTE ON DESMOS: Making tables in Desmos can be helpful and can save you computation time. You are allowed to use it for your homework, just make sure you can also do the calculation with a calculator for the test. Just click on the + symbol in the upper left corner.

2) <u>Graphical Approach</u>



1 Intuitive Definition of a Limi Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

So we might say $\lim_{x \to 2} \frac{x-2}{x^2-4} =$ ________

Caution: The above methods do not always work.

Ex 1 page 52: Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

https://www.desmos.com/calculator/ung82vock4

1) Numerical:

x_1	$f(x_1)$
1	0.16227766
.5	0.16552506
.1	0.1666204
.01	0.1666662
.001	0.16666666

2) Graphical

We need to formalize what we actually mean by limit in order to find ways to compute the value more effectively

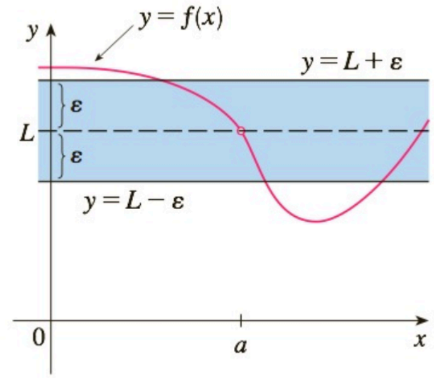
2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

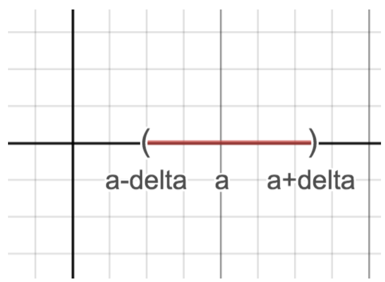
if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Meaning of $|f(x) - L| < \varepsilon$ See horizontal band on graph.



Meaning of $|x - a| < \delta$



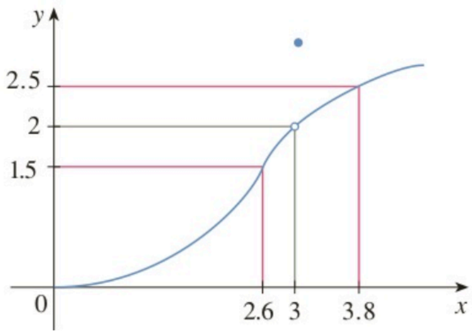
Meaning of $0 < |x - a|$

(See 5A page for additional animations) https://www.desmos.com/calculator/iejhw8zhqd

Example Problems

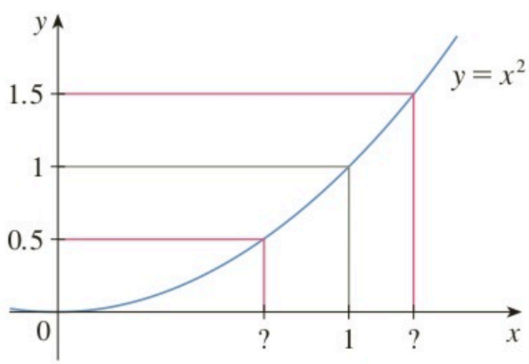
Use the given graph of f to find a number δ such that

if $\quad 0 < |x - 3| < \delta \quad$ then $\quad |f(x) - 2| < 0.5$



Use the given graph of $f(x) = x^2$ to find a number δ such that

if $\quad |x - 1| < \delta \quad$ then $\quad |x^2 - 1| < \frac{1}{2}$



Prove: $\lim_{x\to 1}\left(4x+2\right)=6$

Prove: $\lim_{x\to 3}\left(\frac{x^2+x-12}{x-3}\right)=7$

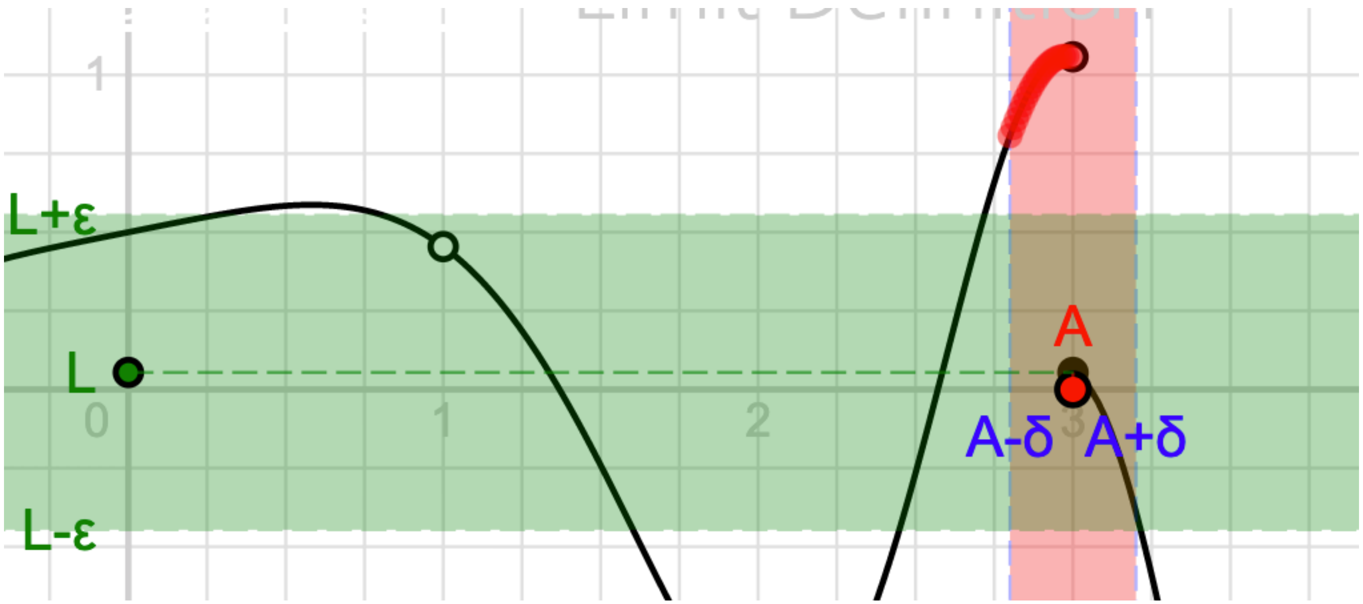
For more challenging problem, see Example 3 page 77.

One Sided Limits

See “Delta-Epsilon Applet” with a=3 on 5A page. What would you say in that example about $\lim_{x \to 3} f(x)$ ______________________

What happens when $x \to 3^{+}$ ______________________

What happens when $x \to 3^{-}$ ______________________

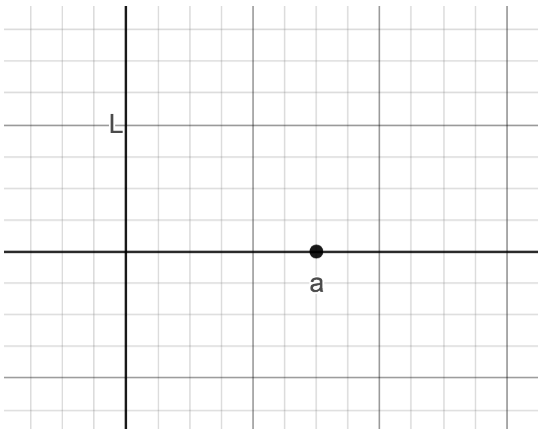


$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Intuitively, what might we mean by $\lim_{x \to a^+} f(x) = L$

Sketch a function showing $\lim_{x \to a^+} f(x) = L$ and create a numerical table which describes the specific example $\lim_{x \to 2^+} f(x) = 5$.

Use these to develop the formal definition.



x_1	$f(x_1)$

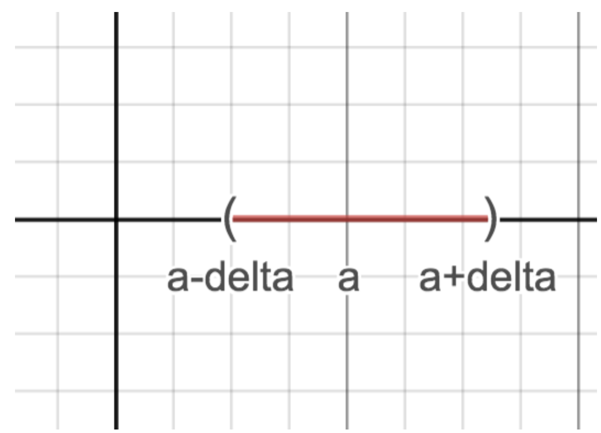
What part of the previous definition is affected if we only consider $x \to a^+$

2 **Precise Definition of a Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $\quad 0 < |x - a| < \delta \quad$ then $\quad |f(x) - L| < \varepsilon$



How do you think the definition would change if we were to look at $x \to a^-$?

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

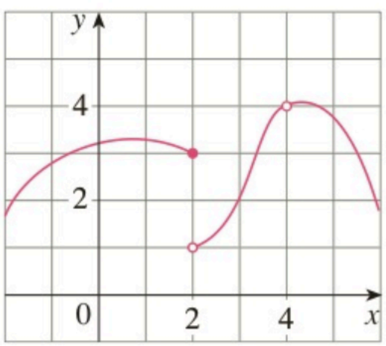
$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Exanple:

Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$

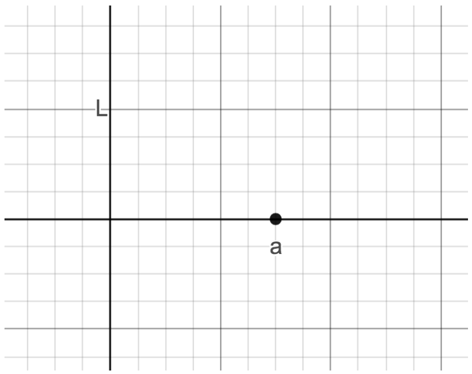


See example 4 page 79 for a proof of a one sided limit.

Infinite Limits/Vertical Asymptotes $\lim_{x \to a} f(x) = \pm\infty$; $\lim_{x \to a^+} f(x) = \pm\infty$; $\lim_{x \to a^-} f(x) = \pm\infty$

Sketch a function such showing $\lim_{x \to a} f(x) = \infty$ and create a numerical table which describes the specific example $\lim_{x \to 2} f(x) = \infty$.

Use these to develop the formal definition



x_1	$f(x_1)$

What part of definition changes?

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Determine the definition for $\lim_{x \to a} f(x) = -\infty$. It may help to sketch a graph or make a table as we have been doing,

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What would be the definition for $\lim_{x \to a^+} f(x) = -\infty$?

Prove $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Computing Infinite Limits at x=a – Vertical Asymptotes

Compute:

$\lim_{x\to 3^+} \frac{2}{x-3} =$ ________ $\lim_{x\to 3^-} \frac{2}{x-3} =$ ________ $\lim_{x\to 3} \frac{2}{x-3} =$ ________

In general, if $\lim_{x\to a} f(x)$ yields an expression of the form $\frac{nonzero\#}{0}$, the graph of $f(x)$ has a vertical asymptote. And

$$\lim_{x\to a} f(x) = \begin{cases} \infty & \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ -\infty & \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ DNE & \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{cases}$$

So we need only determine the SIGN.

EX: Compute

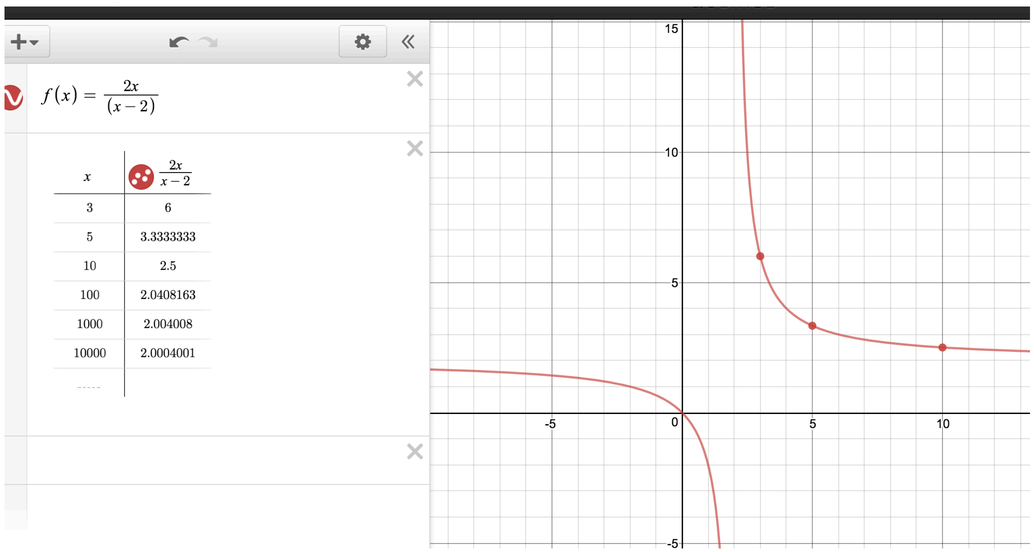
$\lim_{x\to 5^-} \frac{2-x}{(x-5)^2} =$ ________ $\lim_{x\to 5^+} \frac{2-x}{(x-5)^2} =$ ________ $\lim_{x\to 5} \frac{2-x}{(x-5)^2} =$ ________

3.4i Limits at Infinity-Theory

What might this look like graphically? $\lim_{x \to \infty} f(x) = L$

Recall graphing rational functions: $f(x) = \dfrac{2x}{x-2}$ Consider graphical, numerical.

https://www.desmos.com/calculator/dikrtjbb6p



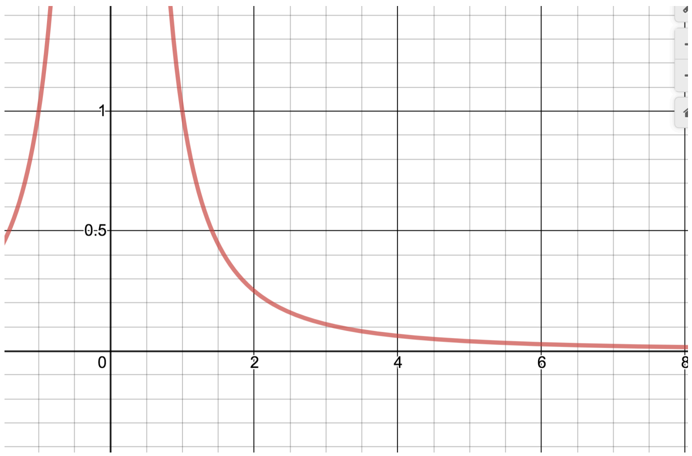
Develop the formal definition for $\lim_{x \to \infty} f(x) = L$

Definition:

Given ______________________________ there must be a corresponding _____________________ such that if _____________________________

then _________________________________.

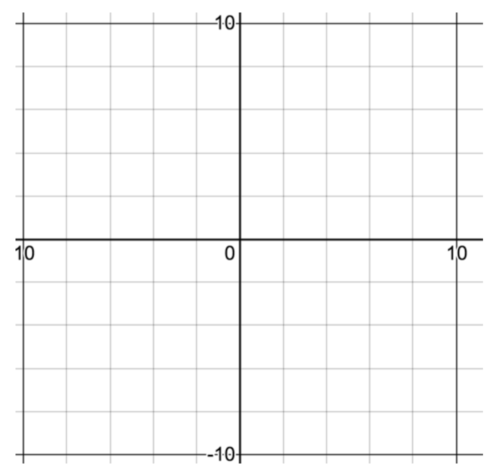
Prove: $\lim_{x\to\infty}\frac{1}{x^2}=0$



x_1	$f(x_1)$
2	0.25
4	0.0625
10	0.01
20	0.0025
100	1×10^{-4}
10000	1×10^{-8}

Other infinite limits: $\lim_{x\to\infty}f(x)=L$ $\lim_{x\to-\infty}f(x)=L$ $\lim_{x\to\infty}f(x)=\pm\infty$ $\lim_{x\to-\infty}f(x)=\pm\infty$

Sketch a function such showing $\lim_{x\to-\infty}f(x)=L$ and develop the formal definition



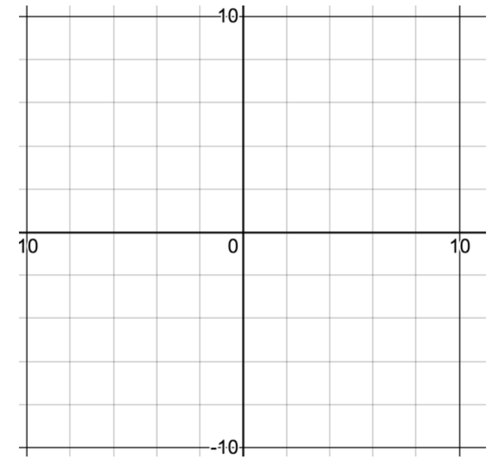
Definition:

Given ______________________________ there must be a corresponding

_____________________ such that if ______________________________ then

__________________________________.

Sketch a function such showing $\lim_{x \to \infty} f(x) = \infty$ and develop the formal definition



Definition:

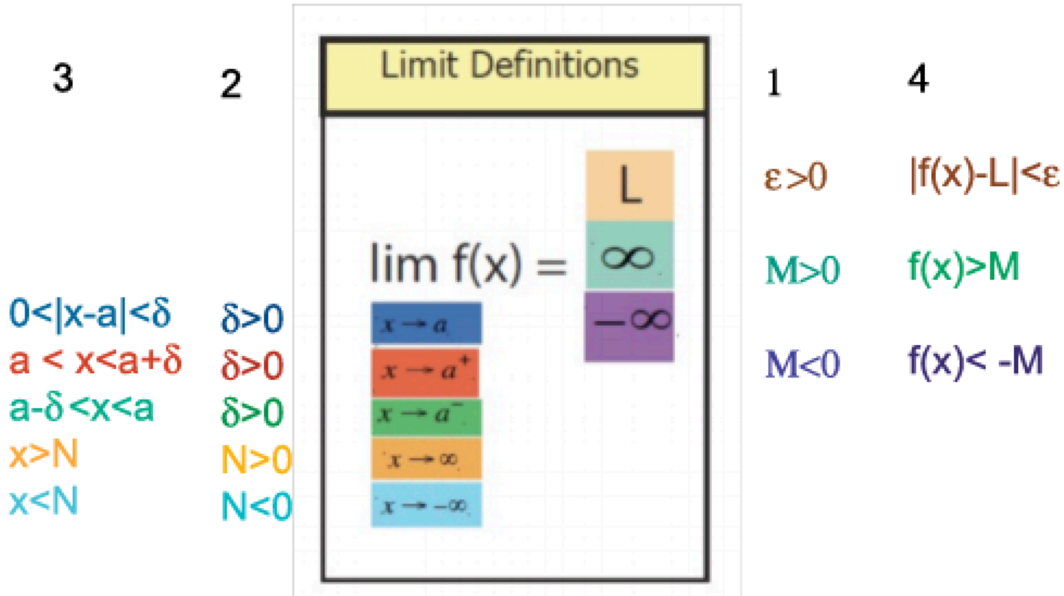
Given ______________________________ there must be a corresponding

_____________________ such that if _____________________________ then

_________________________________.

LIMIT DEFINITIONS SUMMARIZED

Given any ___(1) challenge on f values___, there is a corresponding __(2)__restriction on the x values__ such that if __(3)__ x is in the descibed region________, then ___(4)_____satisfies the challenge__



Try it: What is the definition for $\lim_{x \to -\infty} f(x) = -\infty$

A graph or a table may help. The goal is NOT to memorize, but to understand.

Given any ________________-__, there is a corresponding ________________

such that if _____________, then ___________________________________

1.6 Calculating Limits

So far we have considered two methods of calculating/approximating limits.

1)______________________________ 2)______________________________

Here we seek to find additional methods.

Limit Laws:

7. $\lim_{x \to a} c = c$ **8.** $\lim_{x \to a} x = a$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Examples:

Prove Property 1:

Suppose $\lim_{x \to a} f(x) = L$. This tells us that for every ________there is a ______________ such that if ______________ then ________

Likewise, if and $\lim_{x \to a} g(x) = M$ then for every ________there is a ______________ such that if ______________ then ________

We need to show _________________________, so we must show that

every ________there is a ______________ such that if ______________ then _________________________

Properties 1-5 lead to

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Additional Properties:

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law, which is proved in Section 1.8 as a consequence of Law 10.

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

What do we do in the cases where $\lim_{x \to a} f(x) \neq f(a)$?

$$\lim_{x \to 5} \frac{2x}{x-5}$$

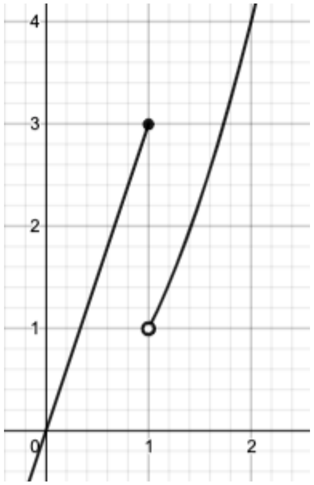
$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$$

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$$

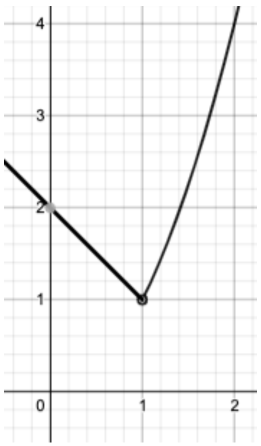
One Sided Limits and piecewise defined functions

Find $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^2 \text{ if } x>1 \\ 3x \text{ if } x\le 1 \end{cases}$



1 Theorem $\lim_{x\to a} f(x)=L$ if and only if $\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$

Find $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^2 \text{ if } x>1 \\ 2-x \text{ if } x\le 1 \end{cases}$



Theorems on Limits

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

Example: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

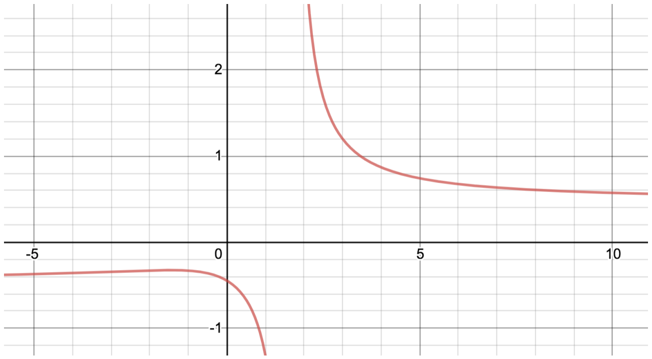
Examples:

$$\lim_{x \to \infty} \frac{2x}{x^2 - 7}$$

$$\lim_{x \to \infty} \frac{5x+3}{2x+9}$$

$$\lim_{x \to \infty} \frac{3x^3}{4x^2 + x + 1}$$

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$



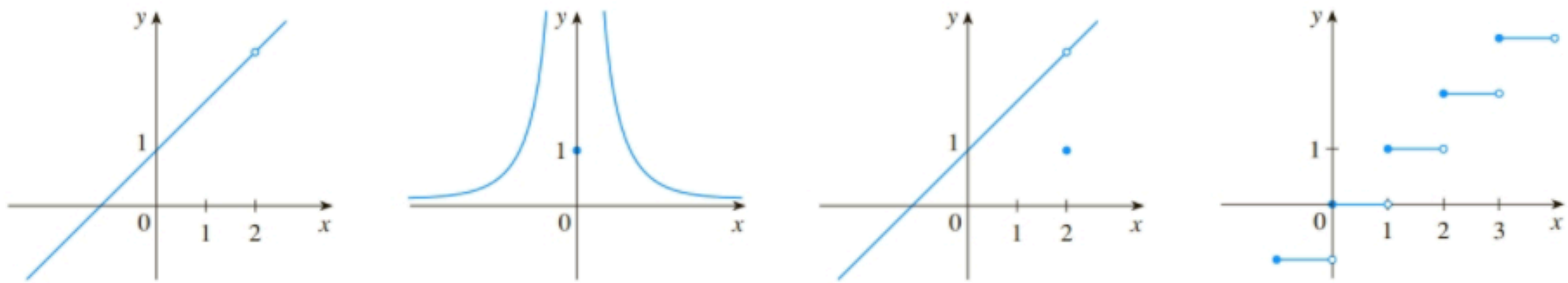
$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

$\lim\limits_{x\to\infty}(x^2+x)$

$\lim\limits_{x\to\infty}(x^2-x)$

1.8 Continuity

Intuitive idea of a continuous function:

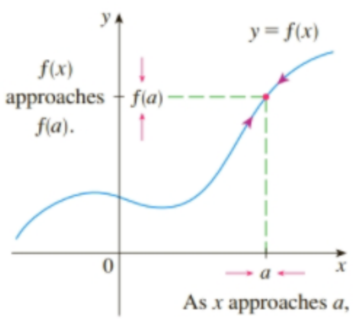


(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$



1 Definition A function f is **continuous at a number** $\boldsymbol{a}$ if

$$\lim_{x \to a} f(x) = f(a)$$

Prove: f(x)=2x+3 is continuous at x=1:

Continuity on an interval: f is said to be continuous on the interval (c,d) if for every $a \in (c,d)$, f(x) is conts at $x = a$

Since we found that for any polynomial or rational function that if $x = a$ is in the domain of f then, $\lim_{x \to a} f(x) = f(a)$

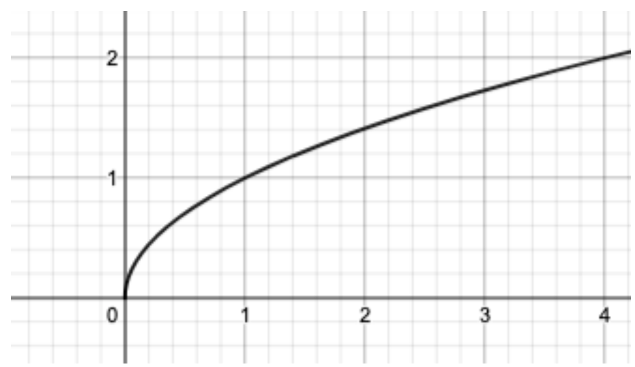
we know that ____

Example: Find the intervals where the following functions are continuous:

$f(x) = 3x^3 - 4x + 2$ $\qquad\qquad$ $g(x) = \dfrac{5x+9}{x+7}$

Continuity at endpoints of an interval

Consider the continuity of $f(x) = \sqrt{x}$



We include endpoints in the interval of continuity in the special case that f is only defined in that interval and one sided continuity is implied.

It turns out that all the basic functions we use are continuous ________________ ____________________ _______________________.

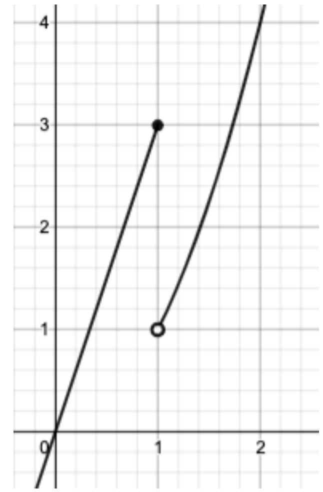
(see book for more details)

<u>Example: Find the intervals where the following functions are continuous:</u>

$f(x) = \cos x + 3$ $\qquad$ $g(x) = \dfrac{3x}{2\sin x - 1}$ $\qquad$ $h(x) = \sqrt{x^2 - x - 12}$

Continuity of Piecewise Defined Functions: (Consider earlier examples)

Find $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^2 \text{ if } x>1 \\ 3x \text{ if } x\le 1 \end{cases}$



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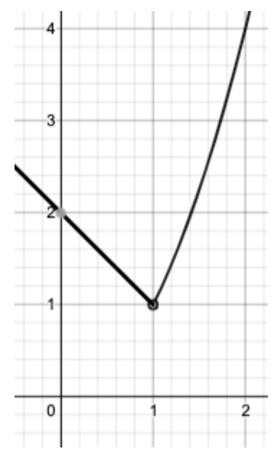
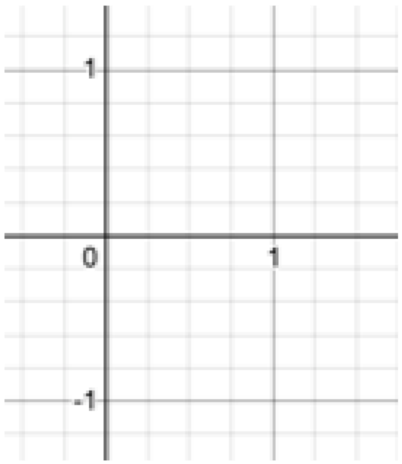
Find $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^2 \text{ if } x>1 \\ 2-x \text{ if } x\le 1 \end{cases}$

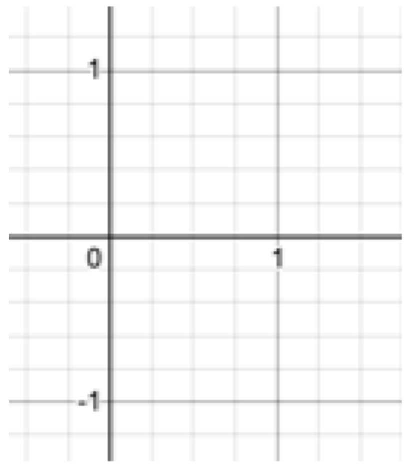


What is so special about continuous functions?

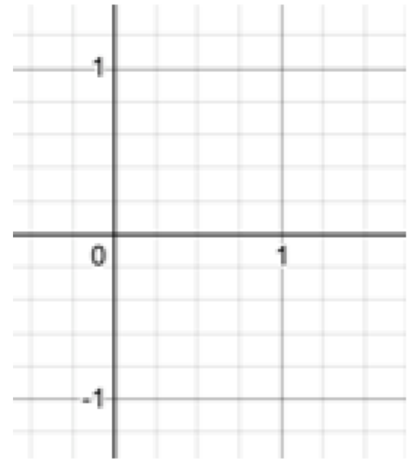
10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



f continuous on [a,b]



f not continuous



not closed interval

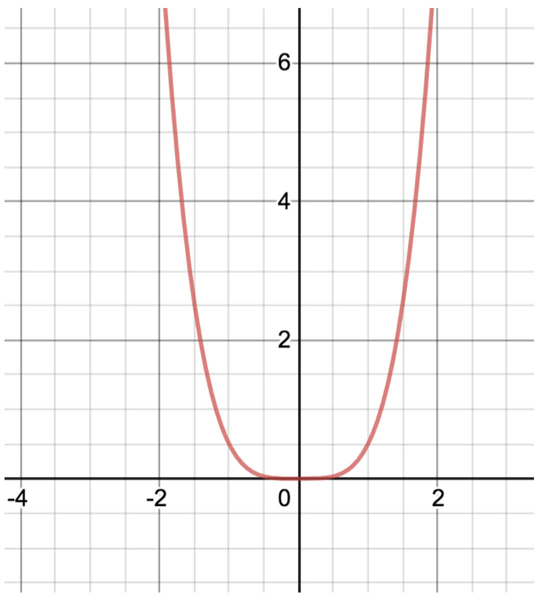
Prove that $x^3 + x - 1 = 0$ has at least one solution,

1.4 and 2.1: Using the limit to compute instantaneous rate of change.
(1) The Tangent Problem and
(2) The Instantaneous Velocity Problems

The Tanget Problem : (Similar to Ex 1 pg 46)

Find the slope of the tangent line to $f(x)=\frac{1}{2}x^4$ at the point $P\left(1,\frac{1}{2}\right)$.

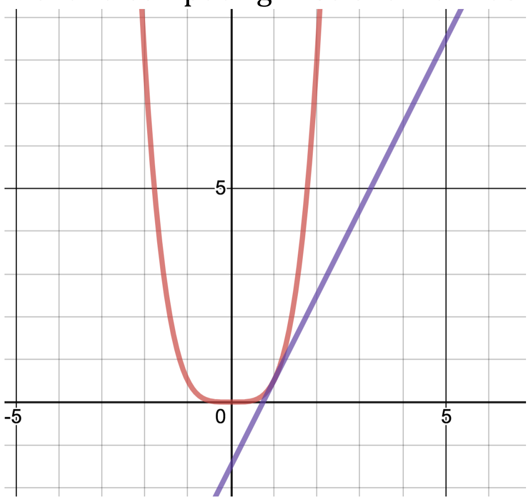
First of all, what do we mean by a tangent line?



Computing slope algebraically, given two points P(x1,y1) and Q(x2,y2) on a line, we compute the slope or the line as __________.
Why wont this work here? So what can we do?

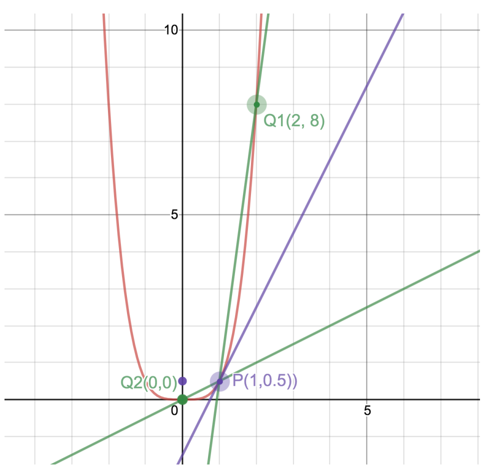
Let's examine 3 approaches.

Method 1) Graphical Approach: We can estimate the slope graphically by drawing a neat graph to scale, drawing the "tangent line" and computing "rise over run" using the LINE sketched. What are possible down sides to this approach?

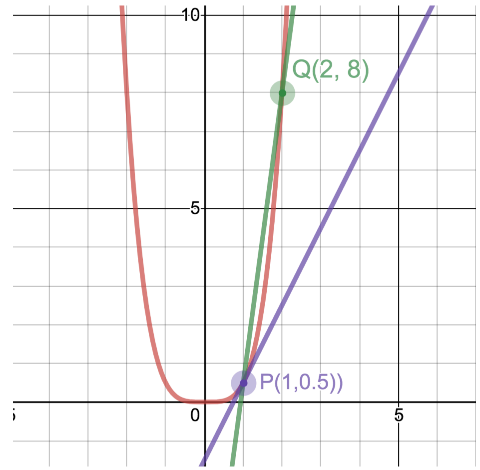


OR

Method 2) Average Approach: A second approach for estimating the slope is to use two different points, Q1 and Q2 on f(x) such that the tangent line lies between the two lines PQ1, PQ2 (so one of PQ1, PQ2 is steeper than our tangent line, one is flatter), find the slopes of PQ1 and of PQ2. Approximate the slope of the tangent line by averaging the two slopes. If we have discrete data, this is usually the approach we use. (see later example)



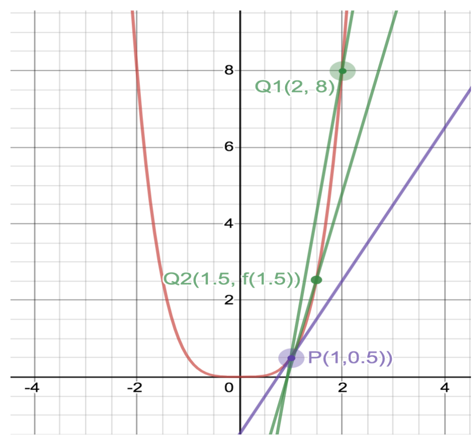
Method 3) "Q approach P " or "limit" approach. We can estimate the slope by introducing a second point and using the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Randomly, let that second point Q(2, 8), a point on f(x). Approximate the slope of the tangent using m_{PQ}



$$m_{\tan} \approx m_{PQ} = \frac{\_ \quad }{\_ \quad} =$$

Do you think that is an overestimate or underestimate. How can we get a better estimate?

Let's choose our second point as a point on f(x) which is closer to P. Let Q be $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, \_\_\_\_\_\right)$ Then



$$m_{\tan} \approx m_{PQ} = \frac{\_ \quad }{\_ \quad} =$$

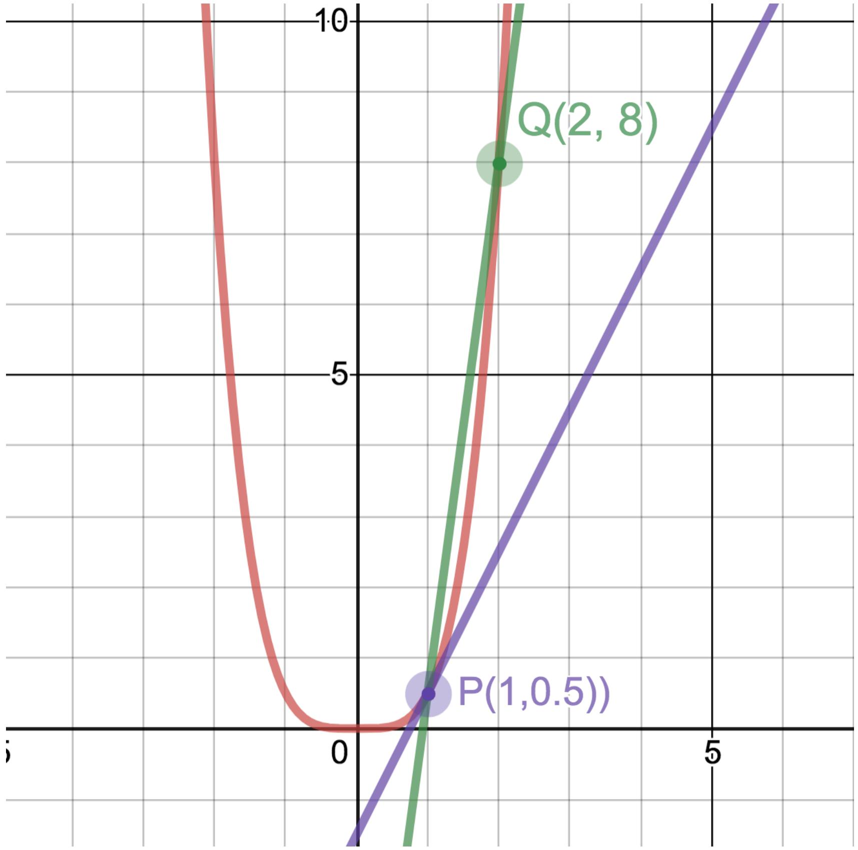
Watch the animation as we continue to let the point Q on f(x) get closer to P.
https://www.desmos.com/calculator/oj5kla60tl (Tangent Secant Desmos on 5A page)

(Do this by sliding the button from h=1 slowly toward h=0)
Notice the how the line PQ (the secant line) more closely approximates the tangent line.

Notice the calculation of slope each time shown on the graph. Do these computations for slope appear to be approaching some value?

The slope computations can also be seen in the table where the x value of Q, (x1) is shown in the left column and m_{PQ} is shown in the right column. Seeing these slope computations in a table may help you answer whether these computations for slope appear to be approaching some value?

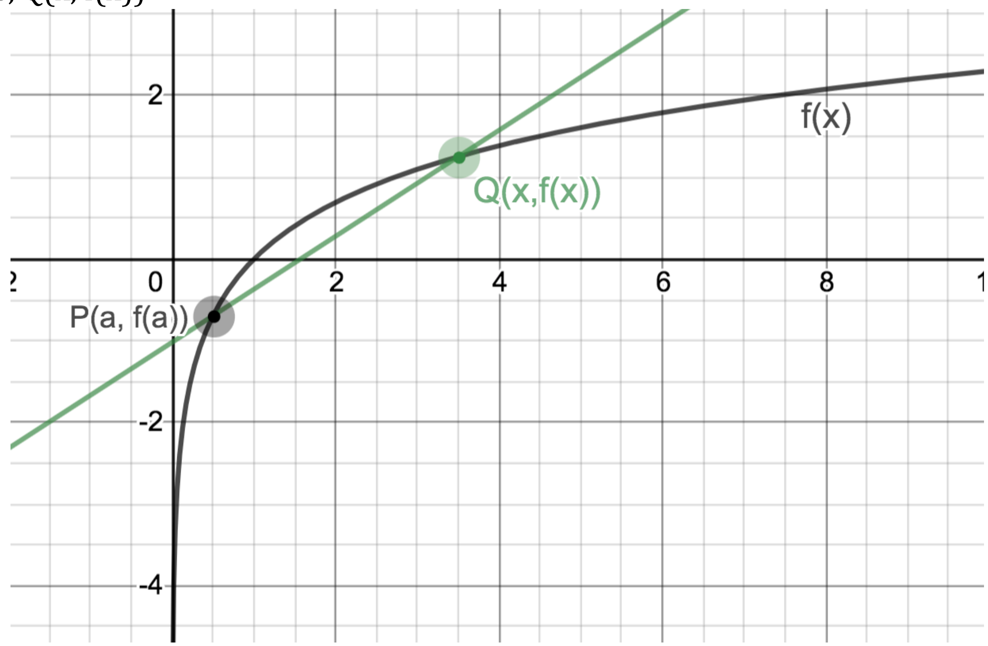
x_1	$f(x_1)$	$m(x_1)$
2	8	7.5
1.5	2.53125	4.0625
1.1	0.73205	2.3205
1.01	0.52030201	2.0302005
1.001	0.502003	2.003002
1.0001	0.50020003	2.0003
1.00001	0.50002	2.00003



If we formalize the process of Method 3 in this example, we see one way we can use limits. Let Q be a general point on f(x) so test Q be $(x, f(x))$ which in this case is $\left(x, \frac{1}{2}x^4\right)$ Then $m_{PQ} = \frac{f(x) - f(1)}{x - 1} = \frac{\quad - \quad}{\quad - \quad}$. In order to let Q move close to P we would need to let x move close to 1. We will write this as

________________________________ And we now know how to compute this limit exactly.

If we apply this process of finding the tangent line to a general function f(x) at some fixed point P(a,f(a)) by introducing a second point Q on the curve, Q(x, f(x))



$$m_{\tan\ at\ x=a} = \lim_{Q \to P} m_{PQ} = \lim \frac{\quad - \quad}{x-a}$$

OR

If we re-label the above graph with point Q being ________________________ , the forumula can equivalently be written as

$$m_{\tan\ at\ x=a} = \lim_{Q \to P} m_{PQ} = \lim \frac{\quad - \quad}{\quad}$$

Note: This is how we *define* the tangent line.

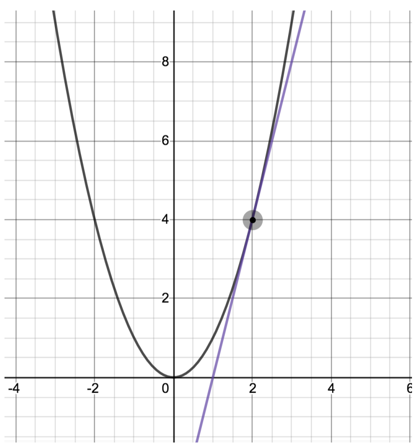
Example: Find the slope of the tangent line to $f(x) = x^2$ at x=2.

Using the first form of the definition:

Two approaches:

1) Put a=2 in at the beginning, then compute limit.

2) Compute limit with "a" and then put in a=2 at the end.

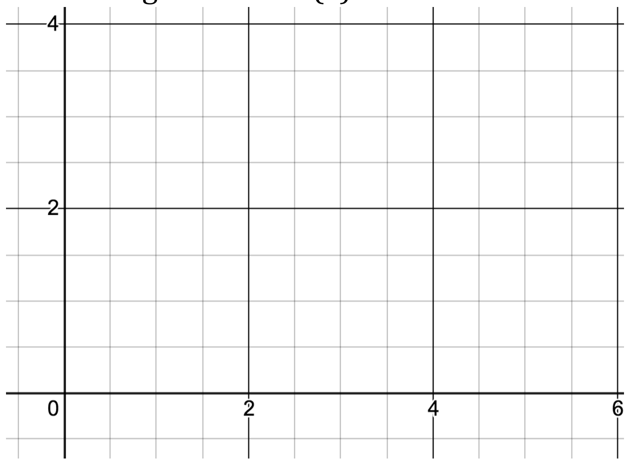


Finding slope of a tangent line if only discrete data is available.

If we are given discrete data (a table as opposed to a formula for f(x)), the limit approach to finding the tangent line exactly is not possible so we use Method 1 or 2 to *estimate* the slope of the tangent line.

Example: Estimate the slope of the tangent line to f(x) at x=3 for the function given.

x_1	$f(x_1)$
0	0
1	1
2	1.4142136
3	1.7320508
4	2
5	2.236068



Method 1

Method 2

The Velocity Problem:

If car travels 20 miles in 30 minutes and we know it is going a constant rate, at any given time the velocity is $r = \frac{d}{t} = \frac{\quad}{\quad} = \_\_\_\_ \, mph$. But what if the rate is NOT constant and we want to know the velocity precisely at the 10th minute? What might we do?

This problem is called finding the instantaneous velocity.

Ex3 pg 49. Given that the distance fallen after t seconds is $s(t)=4.9t^2$ (meters), find the instantaneous velocity at 5 seconds.

First, what is the equation telling us?

What if we want to compute the average velocity over the first two seconds?

Compute the average velocity over the time interval from 4 to 5 seconds.

In general, the Average Velocity over time interval $[t_1, t_2] = \frac{d}{t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$

How do we determine the instantaneous velocity at t=5 seconds?

Smaller time inteval…

Ave Vel on time inteval [4, 5] $= \frac{s(5)-s(4)}{5\ -4} = \frac{122.5-78.4}{1} = 44.1\,m/\sec$

Ave Vel on [4.5, 5] $= \frac{s(5)-s(4.5)}{5\ -4.5} = \frac{122.5-99.225}{0.5} = 46.55\,m/\sec$

Ave Vel on [4.9, 5] $= \frac{s(5)-s(4.9)}{5\ -4.9} = \frac{122.5-117.649}{0.1} = 48.51\,m/\sec$

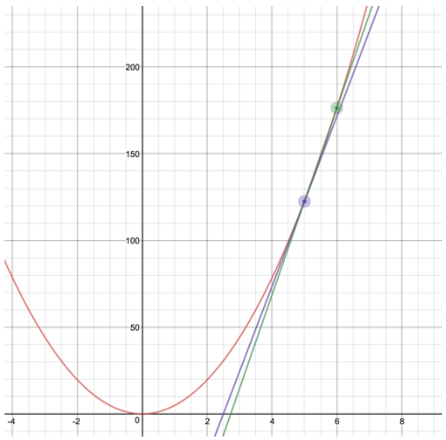
Ave Vel on [4.99, 5] $= \frac{s(5)-s(4.99)}{5\ -4.99} = \frac{122.5-122.01049}{0.01} = 48.951\,m/\sec$

Ave Vel on [4.9999, 5] $= \frac{s(5)-s(4.9999)}{5\ -4.9999} = \frac{122.5-122.459100}{0.0001} = 48.99951\,m/\sec$

See values on page 49 for intervals on the larger side of 5. What do you notice?

Generalizing, the instantaneous velocity for this function at t=5 is $\lim_{t\to 5}\frac{s(t)-s(5)}{t-5} = \lim_{t\to 5}\frac{4.9t^2-122.5}{t-5}$

Show relation to tangent problem.



The problem of finding the tangent line and finding instantaneous velocity, though seemingly physically unrelated are exactly the same process. In fact, any time we seek to find an instantaneous rate of change, we repeat this process. So we introduce a new notation for this process. (For now we just are viewing it as a shorthand notation for this process, we will examine this more in the next unit)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to o} \frac{f(a+h) - f(a)}{h}$$

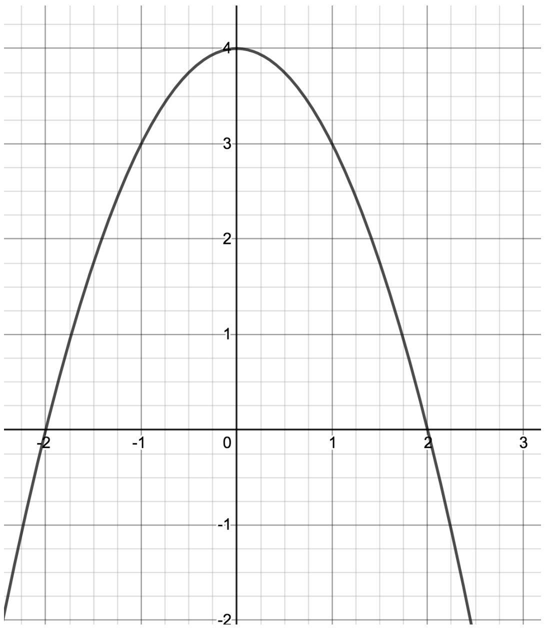
This represents the instantaneous change of f(x) in relation to a change in x when x=a.

Thus, the slope of the tangent to f(x) at x=a can be denoted by $f'(a)$ and the instantaneous velocity of the position function of an object in rectilinear motion s(t) at t=a is

$$s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example: Let $s(t) = \frac{1}{\sqrt{t}}$ be the position (in meters) of an object in rectilinear motion. Find the instantaneous velocity when t=1 second.

Example:



Using the graph of f(x), estimate the following:

(a) $f(1)$ ______________

(b) $f'(1)$ ______________

(c) $f(-1.5)$______________

(d) $f'(-1.5)$______________

Find a value of "c" such that

(e) $f(c) = 0$______________

(f) $f'(c) = 0$______________

Example:

The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) Is $f'(8)$ positive or negative? Explain.